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The Euler Approach and Direct Tensor Calculus in the Problem of the Physical Nature of the Coriolis Effects

M. E. Podolsky and S. V. Cherenkova*
St. Petersburg State Marine Technical University, St. Petersburg, 190121 Russia
**e-mail: svchpar@list.ru*

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Abstract—The problem of the Coriolis inertia force moment is considered in spatial formulation in the framework of studying the hydrodynamics of moving channels. The physical meaning of the obtained formulas is elucidated using direct tensor calculus and the Euler approach to the description of kinematics. The results provide a basis for extending the well-known Euler's turbine equation to the general case of spatial motion and refining the conditions of applicability of the Gauss–Ostrogradsky formula.

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Coriolis acceleration $\mathbf{a}_c$ and Coriolis inertia force moment $\mathbf{M}_c$ are defined by the well-known formulas

$$\mathbf{a}_c = 2\boldsymbol{\omega} \times \mathbf{w}, \quad \mathbf{M}_c = -m\mathbf{r} \times \mathbf{a}_c, \tag{1}$$

where m, $\mathbf{r}$, and $\mathbf{w}$ are the mass, radius vector, and relative velocity vector of a material point and $\boldsymbol{\omega}$ is the angular velocity vector.

Recently, it was shown [1] that the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + \mathbf{b} \times (\mathbf{a} \times \mathbf{c}) \tag{2}$$

implies the following formula:

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = -\mathbf{u} \times \mathbf{w} + \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{w}), \tag{3}$$

where

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r} \tag{4}$$

is the bulk velocity.

In addition, it was shown [1] that

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = \mathbf{u} \times \mathbf{w} + \mathbf{w}^* \frac{d(\mathbf{r} \times \mathbf{u})}{d\mathbf{r}}. \tag{5}$$

Below, we present another way (different from that used in [1]) to prove relation (5). Taking into account that

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = (\mathbf{w} \times \boldsymbol{\omega}) \times \mathbf{r} \quad \text{and} \quad \mathbf{w} = \mathbf{w}^* E, \tag{6}$$

where E is the unit tensor, we can write

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = [(\mathbf{w}^* E) \times \boldsymbol{\omega}] \times \mathbf{r} = \mathbf{w}^*[(E \times \boldsymbol{\omega}] \times \mathbf{r}]. \tag{7}$$

Since $\boldsymbol{\omega}$ is independent of $\mathbf{r}$ and (with allowance for formula (3))

$$\frac{d\mathbf{u}}{d\mathbf{r}} = \frac{d}{d\mathbf{r}}(\boldsymbol{\omega} \times \mathbf{r}) = -E \times \boldsymbol{\omega}, \tag{8}$$

we can rewrite Eq. (7) as follows:

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = -\mathbf{w}\frac{d\mathbf{u}}{d\mathbf{r}} \times \mathbf{r}.$$

Then, taking into account that [2]

$$\frac{d\mathbf{u}}{d\mathbf{r}} \times \mathbf{r} = \frac{d(\mathbf{u} \times \mathbf{r})}{d\mathbf{r}} + \frac{d\mathbf{r}}{d\mathbf{r}} \times \mathbf{u} = \frac{d(\mathbf{u} \times \mathbf{r})}{d\mathbf{r}} + E \times \mathbf{u},$$

we obtain

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = -\mathbf{w}^* \frac{d(\mathbf{u} \times \mathbf{r})}{d\mathbf{r}} - \mathbf{w}^* E \times \mathbf{u},$$

and, hence,

$$\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{w}) = \mathbf{w}^* \frac{d(\mathbf{r} \times \mathbf{u})}{d\mathbf{r}} + \mathbf{u} \times \mathbf{w},$$

which coincides with relation (5).

Equations (1), (3), and (5) yield the following formula (see also [1]):

$$\mathbf{M}_c = -m\left[\boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{w}) + \mathbf{w}^* \frac{d(\mathbf{r} \times \mathbf{u})}{d\mathbf{r}}\right], \tag{9}$$

the right-hand side of which is a sum of two terms. Note that the right-hand side of the formula for $\mathbf{a}_c$ in Eqs. (1) can also be considered as a sum of two terms, but these terms are equal, in contrast to those entering into formula (9).

The physical (more precisely, kinematical) nature of $\boldsymbol{\omega} \times \mathbf{w}$ product is well known. The appearance of this term in the formula for $\mathbf{a}_c$ is explained by two factors. First, the rotation of a body is accompanied by variation of the angular orientation of the line of action of the velocity vector; second, the passage of a point over the body is accompanied by variation of the bulk velocity.

Based on the Euler approach, it was recently shown [3, 4] that the two factors can be taken into account, in addition to using formula (1), by the following equation:

$$\mathbf{a}_c = \mathbf{u}^* \frac{d\mathbf{w}}{d\mathbf{r}} + \mathbf{w}^* \frac{d\mathbf{u}}{d\mathbf{r}}, \tag{10}$$

where the first term reflects the influence of the first factor and the second term corresponds to the second factor.

By analogy with Eq. (10), we can also obtain a formula for moment $\mathbf{M}_c$. We proceed from the fact that the moment of inertia forces is the time derivative (with a minus sign) of the kinetic momentum. The appearance of $\mathbf{M}_c$ is explained by the same factors as the appearance of Coriolis acceleration $\mathbf{a}_c$, so that the formula for $\mathbf{M}_c$ can be written as follows:

$$\mathbf{M}_c = -m\left[\mathbf{u}*\frac{d(\mathbf{r}\times\mathbf{w})}{d\mathbf{r}} + \mathbf{w}*\frac{d(\mathbf{r}\times\mathbf{u})}{d\mathbf{r}}\right]. \qquad (11)$$

Taking into account that, as was shown in [3, 4] for the problem under consideration,

$$\mathbf{u}*\frac{d(\mathbf{r}\times\mathbf{w})}{d\mathbf{r}} = \boldsymbol{\omega}\times(\mathbf{r}\times\mathbf{w}),$$

we conclude that formula (11) fully coincides with (9).

The coincidence of formulas (11) and (9) confirms the validity of physical concepts used to derive formula (11) and, in addition, it is evidence of the efficiency of this approach, since the second term in formula (11) was written directly, without using the rather artificial transformations employed in the derivation of relation (5).

The fact that the second term in formula (11) contains a scalar product of velocity $\mathbf{w}$ and derivative $d(\mathbf{r}\times\mathbf{u})/(d\mathbf{r})$ is of fundamental significance in that it makes the subsequent transformations possible, as a result of which the main equation of the theory of turbomachines (Euler's turbine equation) can be extended to the general case of spatial motion.

In connection with this, it should also be noted that an analysis of formula (11) revealed (as will be reported in a subsequent publication) a paradoxical phenomenon related to the conditions of applicability of a tensor analog of the Gauss–Ostrogradsky formula. The importance of this result is determined by the fact that it allows one to avoid serious errors in hydrodynamic calculations of moving channels.

In conclusion, it is also important to draw attention to one circumstance—the role that was played by a combination of the Euler approach and direct tensor calculus methods in elucidating the physical meaning of the Coriolis effects.

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